Triangulated Categories V: Glueing t-structures

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1 Introduction

This exposition is the fifth part of our study of triangulated categories in algebraic geometry. We shall explain how to glue $t$-structures on triangulated categories and describe their proofs in detail. Main reference is [BBD].

2 $t$-exact functors

Definition 2.1. Let $C, D$ be triangulated categories and let $F : C \to D$ be an exact functor. Let $(C^{\leq 0}, C^{> 0}), (D^{\leq 0}, D^{> 0})$ be $t$-structures on $C, D$, respectively. We say that $F$ is $t$-exact if $F C^{\leq 0} \subseteq D^{\geq 0}$ and $F C^{> 0} \subseteq D^{< 0}$. In the case, the $t$-structures $(C^{\leq 0}, C^{> 0})$ and $(D^{\leq 0}, D^{> 0})$ are said to be compatible.

Proposition 2.2. Let $C, D$ and $F$ be as in Definition 2.1. Let $(C^{\leq 0}, C^{> 0}), (D^{\leq 0}, D^{> 0})$ be $t$-structures on $C, D$, respectively. Let $\tau_{\leq n} C, \tau_{> n} C$ (resp. $\tau_{\leq n} D, \tau_{> n} D$) be truncation functors for each $n \in \mathbb{Z}$ with respect to the $t$-structure of $C$ (resp. $D$). If $F : C \to D$ is $t$-exact, then we have

$$F \circ \tau_{\leq n} C \cong \tau_{\leq n} D \circ F \quad \text{and} \quad F \circ \tau_{> n} C \cong \tau_{> n} D \circ F$$

for each $n \in \mathbb{Z}$.

Proof. Let $X$ be any object of $C$. For any $n \in \mathbb{Z}$, we have the distinguished triangles

$$\tau_{\leq n} X \to X \to \tau_{> n+1} X \to \tau_{\leq n} X$$

and

$$F(\tau_{\leq n} X) \to F(X) \to F(\tau_{> n+1} X) \to F(\tau_{\leq n} X).$$
By Proposition 2.9 of [Ter 15], we obtain $F(\tau_{\leq a}X) \simeq \tau_{\leq a}F(X)$ and $F(\tau_{\geq a}X) \simeq \tau_{\leq a}F(X)$.

**Proposition 2.3.** Let $(C^{\leq 0}, C^{\geq 0})$ and $(D^{\leq 0}, D^{\geq 0})$ be t-structures on triangulated categories $C$ and $D$, respectively. Let $F : C \rightarrow D$ be a fully faithful exact functor. Assume that $F$ is a t-exact functor. Then we have the following.

(i) If $F(X) \in D^{\leq 0}$ for $X \in C$, then $X \in C^{\leq 0}$, i.e., $F(C^{\leq 0}) = F(C \cap D^{\leq 0})$.

(ii) If $F(X) \in D^{\geq 0}$ for $X \in C$, then $X \in C^{\geq 0}$, i.e., $F(C^{\geq 0}) = F(C \cap D^{\geq 0})$.

**Proof.** (i) If $F(X) \in D^{\leq 0}$ for $X \in C$, then for any $Y \in C^{\geq 0}$, $\text{Hom}_C(X, Y) = \text{Hom}_D(F(X), F(Y)) = 0$ since $F(Y) \in D^{\geq 0}$. Hence we have $X \in (C^{\leq 0}) = C^{\leq 0}$ by Proposition 2.6 of [Ter 15].

(ii) If $F(X) \in D^{\geq 0}$ for $X \in C$, then for any $Y \in C^{\leq 0}$, $\text{Hom}_C(Y, X) = \text{Hom}_D(F(Y), F(X)) = 0$ since $F(Y) \in D^{\leq 0}$. Hence we have $X \in (C^{\geq 0}) = C^{\geq 0}$ by Proposition 2.6 of [Ter 15].

**Corollary 2.4.** Let $\mathcal{D}$ be a triangulated category and let $\mathcal{C}$ be a full triangulated subcategory of $\mathcal{D}$. Assume that there exist t-structures $(C^{\leq 0}, C^{\geq 0})$, $(D^{\leq 0}, D^{\geq 0})$ on $\mathcal{C}$, $\mathcal{D}$, respectively. If the inclusion functor $\tau : \mathcal{C} \hookrightarrow \mathcal{D}$ is t-exact, then a t-structure on $\mathcal{D}$ determines a unique compatible t-structure on $\mathcal{C}$.

**Proposition 2.5.** Let $(\mathcal{D}, T)$ be a triangulated category and let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t-structure on $\mathcal{D}$. Assume that $\mathcal{C}$ is a strictly full triangulated subcategory of $\mathcal{D}$. Then the pair

$$(C^{\leq 0}, C^{\geq 0}) := (C \cap D^{\leq 0}, C \cap D^{\geq 0})$$

is a t-structure on $\mathcal{C}$ if and only if $\mathcal{C}$ is stable under the functor $\tau_{C^{\leq 0}}$, i.e., $\tau_{C^{\leq 0}} \mathcal{C} \subset \mathcal{C}$. In this case, the t-structure $(C^{\leq 0}, C^{\geq 0})$ on $\mathcal{C}$ is called the *induced t-structure*.

**Proof.** If $(C^{\leq 0}, C^{\geq 0})$ is a t-structure on $\mathcal{C}$, then $\tau$ is t-exact. By Proposition 2.2, we obtain $\tau_{C^{\leq 0}} X \simeq \tau_{C^{\leq 0}} X \in C^{c^{\leq 0}}$ for any $X \in \mathcal{C}$. Hence $\tau_{C^{\leq 0}} \mathcal{C} \subset \mathcal{C}$.

Conversely, assume that $\tau_{C^{\leq 0}} \mathcal{C} \subset \mathcal{C}$. Let us check the conditions of Definition 2.1 of [OT 14] to show that the pair $(C^{\leq 0}, C^{\geq 0})$ is a t-structure on $\mathcal{C}$.

(a) We have

$$T C^{\leq 0} = T(C \cap D^{\leq 0}) \subset C \cap T D^{\leq 0} \subset C \cap D^{\leq 0} = C^{\leq 0}$$

and

$$T^{-1} C^{\geq 0} = T^{-1}(C \cap D^{\geq 0}) \subset C \cap T^{-1} D^{\geq 0} \subset C \cap D^{\geq 0} = C^{\geq 0}.$$ (b) is clear because $C^{\leq 0} \subset D^{\leq 0}$ and $T^{-1} C^{\geq 0} \subset T^{-1} D^{\geq 0}$.

(c) For each $X \in \mathcal{C}$, we have a distinguished triangle

$$\tau_{C^{\leq 0}} X \rightarrow X \rightarrow \tau_{C^{\geq 0}} X \rightarrow T \tau_{C^{\leq 0}} X$$

in $\mathcal{D}$. If $\tau_{C^{\leq 0}} \mathcal{C} \subset \mathcal{C}$, then $\tau_{C^{\leq 0}} X \in \mathcal{C} \cap D^{\leq 0} = C^{\leq 0}$. By Corollary 3.19 of [OT 14], we see that $\tau_{C^{\leq 0}} X$ is isomorphic to an object of $\mathcal{C}$, and thus it belongs to $\mathcal{C}$, because $\mathcal{C}$ is a strictly full subcategory of $\mathcal{D}$. Hence $\tau_{C^{\leq 0}} X \in \mathcal{C} \cap D^{\leq 1} = C^{\leq 1}$. Therefore by Corollary 3.18 of [OT 14], we obtain a decomposition in $\mathcal{C}$.
3 Gluing t-structures

Let $\mathcal{D}$, $\mathcal{A}$ and $\mathcal{B}$ be triangulated categories and let $F: \mathcal{A} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{B}$ be exact functors. We suppose the following conditions:

GL 1 $F$ admits a left adjoint functor $F_L$ and a right adjoint functor $F_R: F_L \dashv F \dashv F_R$.

GL 2 $G$ admits a left adjoint functor $G_L$ and a right adjoint functor $G_R: G_L \dashv G \dashv G_R$.

GL 3 $G \circ F = 0$.

GL 4 There exist the following two distinguished triangles:

\[
\begin{align*}
&G_L GX \to X \to F F_L X \to TG_L GX, \\
&FF_R X \to X \to G_R GX \xrightarrow{d'} TFF_R X.
\end{align*}
\]

GL 5 $F$, $G_L$ and $G_R$ are fully faithful.

**Proposition 3.1.** From the above conditions, we obtain the following.

(i) $F_L \circ G_L = 0$ and $F_R \circ G_R = 0$.

(ii) The morphisms $d$ and $d'$ are unique.

(iii) The units and counits of the adjunctions are isomorphisms:

\[
F_L \circ F \xrightarrow{\sim} \text{id}_A \xrightarrow{\sim} F_R \circ F \quad \text{and} \quad G \circ G_R \xrightarrow{\sim} \text{id}_B \xrightarrow{\sim} G \circ G_L.
\]

**Proof.** (i) For any $Y \in \mathcal{A}$, we have

\[
\text{Hom}_A(F_L G_L X, Y) \cong \text{Hom}_D(G_L X, FY) \cong \text{Hom}_B(X, GFY) = 0.
\]

For any $X \in \mathcal{A}$, we have

\[
\text{Hom}_A(X, F_R G_R Y) \cong \text{Hom}_D(FX, G_R Y) \cong \text{Hom}_B(GFX, Y) = 0.
\]

By the Yoneda Lemma, we obtain $F_L \circ G_L = 0$ and $F_R \circ G_R = 0$.

(ii) In the distinguished triangle (3.1), we have

\[
\text{Hom}_D(TG_L GX, FF_L X) \cong \text{Hom}_D(G_L TG_X, FF_L X)
\]

\[
\cong \text{Hom}_D(TGX, GFF_L X)
\]

\[
= 0.
\]

In the distinguished triangle (3.2), we have

\[
\text{Hom}_D(TFF_R X, G_R GX) \cong \text{Hom}_D(FTFF_R, G_R GX)
\]

\[
\cong \text{Hom}_D(GFTF_R X, GX)
\]

\[
= 0.
\]

By Lemma 2.8 of [Ter 15], we see that $d$ and $d'$ are unique.

(iii) This follows from Proposition 2.8 of [Ter 16].

**Theorem 3.2.** Let $(\mathcal{A}^{<0}, \mathcal{A}^{\leq 0})$ be a $t$-structure on $\mathcal{A}$ and let $(\mathcal{B}^{<0}, \mathcal{B}^{\leq 0})$ be a $t$-structure on $\mathcal{B}$. We define the strictly full subcategories $\mathcal{D}^{<0}$ and $\mathcal{D}^{\geq 0}$ as follows.

\[
\mathcal{D}^{<0} := \{ X \in \mathcal{D} \mid F_L X \in \mathcal{A}^{<0}, GX \in \mathcal{B}^{<0} \},
\]

\[
\mathcal{D}^{\geq 0} := \{ X \in \mathcal{D} \mid F_R X \in \mathcal{A}^{\geq 0}, GX \in \mathcal{B}^{\geq 0} \}.
\]

Then the pair $(\mathcal{D}^{<0}, \mathcal{D}^{\geq 0})$ is a $t$-structure on $\mathcal{D}$. 

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Proof. We show that the conditions TS 1—TS 3 in Definition 2.1 of [Ter 15] are satisfied.

(TS 1) We have

\[
T^0 \mathcal{D} = \{ TX \mid X \in \mathcal{D}, F_L X \in \mathcal{A}^{S0}, \, GX \in \mathcal{B}^{S0}\} \\
= \{ Y \in \mathcal{D} \mid F_L T^{-1} Y \in \mathcal{A}^{S0}, \, GT^{-1} Y \in \mathcal{B}^{S0}\} \\
= \{ Y \in \mathcal{D} \mid T^{-1} F_L Y \in \mathcal{T} \mathcal{A}^{S0}, \, T^{-1} G Y \in \mathcal{T} \mathcal{B}^{S0}\} \\
= \{ Y \in \mathcal{D} \mid F_L Y \in T \mathcal{A}^{S0}, \, G Y \in T \mathcal{B}^{S0}\} \\
\subseteq \{ Y \in \mathcal{D} \mid F_L Y \in \mathcal{A}^{S0}, \, G Y \in \mathcal{B}^{S0}\} \\
= \mathcal{D}^{S0}
\]

and

\[
T^{-1} \mathcal{D}^{S0} = \{ T^{-1} X \mid X \in \mathcal{D}, F_R X \in \mathcal{A}^{S0}, \, GX \in \mathcal{B}^{S0}\} \\
= \{ Y \in \mathcal{D} \mid F_R T Y \in \mathcal{A}^{S0}, \, GT Y \in \mathcal{B}^{S0}\} \\
= \{ Y \in \mathcal{D} \mid T F_R Y \in \mathcal{T} \mathcal{A}^{S0}, \, T G Y \in \mathcal{T} \mathcal{B}^{S0}\} \\
= \{ Y \in \mathcal{D} \mid F_R Y \in T^{-1} \mathcal{A}^{S0}, \, G Y \in T^{-1} \mathcal{B}^{S0}\} \\
\subseteq \{ Y \in \mathcal{D} \mid F_R Y \in \mathcal{A}^{S0}, \, G Y \in \mathcal{B}^{S0}\} \\
= \mathcal{D}^{S0}
\]

(TS 2) From the distinguished triangle (3.1), we obtain a long exact sequence

\[\ldots \to \text{Hom}_\mathcal{D}(F L X, Y) \to \text{Hom}_\mathcal{D}(X, Y) \to \text{Hom}_\mathcal{D}(G L GX, Y) \to \ldots.\]

If \(X \in \mathcal{D}^{S0}\), then \(F_L X \in \mathcal{A}^{S0}\) and \(GX \in \mathcal{B}^{S0}\). If \(Y \in T^{-1} \mathcal{D}^{S0}\), then \(F_R TY \in \mathcal{A}^{S0}\) and \(GT Y \in \mathcal{B}^{S0}\), and thus \(T F_R Y \in \mathcal{A}^{S0}\) and \(T G Y \in \mathcal{B}^{S0}\). Hence we obtain \(F_R Y \in T^{-1} \mathcal{A}^{S0}\) and \(G Y \in T^{-1} \mathcal{B}^{S0}\).

Then we have

\[
\text{Hom}_\mathcal{D}(F L X, Y) \simeq \text{Hom}_A(F_L X, F_R Y) = 0
\]

and

\[
\text{Hom}_\mathcal{D}(G L GX, Y) \simeq \text{Hom}_B(G X, G Y) = 0.
\]

Therefore \(\text{Hom}_\mathcal{D}(X, Y) = 0\) for any \(X \in \mathcal{D}^{S0}\) and \(Y \in T^{-1} \mathcal{D}^{S0}\).

(TS 3) Let \(X \in \mathcal{D}\). From Proposition 2.9. (iii) of [Ter 15], we have a distinguished triangle in \(\mathcal{B}\)

\[
\tau_{\leq 0}^B(G X) \xrightarrow{\delta_{B}(G X)} G X \xrightarrow{\varepsilon_{B}(G X)} \tau_{\geq 1}^B(G X) \xrightarrow{d_{B}^1(G X)} T \tau_{\leq 0}^B(G X).
\]

Hence we obtain a distinguished triangle

\[
Y \xrightarrow{\delta_Y} X \xrightarrow{G \delta} G Y \xrightarrow{\varepsilon_{G}(Y)} \tau_{\geq 1}^A(F L Y) \xrightarrow{d_{G}^1(Y)} T \tau_{\leq 0}^A(F L Y).
\]

where \(\varepsilon_Y : \text{id}_\mathcal{D} \to G \circ F_L\) is the counit of the adjunction.

Similarly, we have a distinguished triangle in \(\mathcal{A}\)

\[
\tau_{\leq 0}^A(F L Y) \xrightarrow{\varepsilon_{F}(Y)} F L Y \xrightarrow{d_{F}(F L Y)} \tau_{\geq 1}^A(F L Y) \xrightarrow{d_{F}^1(Y)} T \tau_{\leq 0}^A(F L Y).
\]

Hence we obtain a distinguished triangle

\[
A \xrightarrow{\varepsilon_{F}} Y \xrightarrow{F \delta} F L Y \xrightarrow{T \varepsilon_{F}(F L Y)} T A,
\]

where \(\varepsilon_{F} : \text{id}_\mathcal{D} \to F \circ F_L\) is the counit of the adjunction.

Then we have the octahedron diagram for distinguished triangles (3.4) and (3.6)
Applying $F_l$ to the distinguished triangle (3.6), we have

(3.7) \[ F_l(A) \xrightarrow{F_l(f)} F_l(Y) \xrightarrow{F_l(f)(\varepsilon(F_l(Y)))} F_l(F \tau_{\geq 1} F_l(Y)) \rightarrow TF_l(A). \]

From Proposition 2.4, (i) of [Ter 16], we see that $F_l (F (\varepsilon(F_l(Y))) \circ \varepsilon(Y))$ factors as follows:

Hence the distinguished triangle (3.7) is isomorphic to the distinguished triangle (3.5). Therefore by Proposition 2.9. (iii) of [Ter 15], we obtain $F_l(A) \simeq_{\tau_{\geq 1}} F_l(Y) \in \mathcal{A}_{\geq 0}$, and thus $F_l(A) \in \mathcal{A}^{>0}$.

Applying $G$ to the distinguished triangle of the forth row in the above octahedron diagram, we have

(3.8) \[ G \xrightarrow{G \circ F} \gamma \xrightarrow{G(G \circ \varepsilon(GX))} G \gamma \xrightarrow{G \gamma} \tau_{\geq 1} \gamma. \]

Hence the distinguished triangle (3.8) is isomorphic to the distinguished triangle (3.3). Therefore by Proposition 2.9. (iii) of [Ter 15], we obtain $G \gamma \in \mathcal{B}^{>0}$. Moreover, applying $G$ to the distinguished triangle (3.6), it follows from $G \circ F = 0$ that $GA \simeq GY \in \mathcal{B}^{>0}$, and thus $GA \in \mathcal{B}^{>0}$.

Applying $F_k$ to the distinguished triangle of the forth row in the above octahedron diagram, we have
\[ F_\star \tau_{\mathcal{A}} \mathcal{F}_L Y \Rightarrow F_\star B \Rightarrow F_\star G \tau_{\mathcal{B}} \mathcal{G} X \Rightarrow T F_\star \tau_{\mathcal{A}} \mathcal{F}_L Y. \]

Since \( F_\star \circ G_\star = 0 \) and \( \epsilon_F : \text{id}_\mathcal{A} \Rightarrow F_\star \circ F \), we obtain \( F_\star B \simeq \tau_{\mathcal{A}} \mathcal{F}_L Y \in \mathcal{A}^{\geq 1} \Rightarrow T^{-1} \mathcal{A}^{\leq 0} \), and thus \( F_\star B \in T^{-1} \mathcal{A}^{\leq 0} \).

Therefore, for any \( X \in \mathcal{D} \), we have a distinguished triangle
\[ A \rightarrow X \rightarrow B \rightarrow TA \]
with \( A \in \mathcal{D}^{\leq 0} \) and \( B \in T^{-1} \mathcal{D}^{\geq 0} \).

**Remark 3.3.** Let \( X \in \mathcal{D} \). Under the above conditions, we have the following:
\[ F_\star X \in \mathcal{A}^{\geq 0} \iff F_\star X \in \prod (\mathcal{A}^{-1})_\mathcal{A} \]
\[ \iff \text{Hom}_\mathcal{A} (F_\star X, Y) = 0 \text{ for all } Y \in \mathcal{A}^{-1} \]
\[ \iff \text{Hom}_\mathcal{D} (X, F Y) = 0 \text{ for all } Y \in \mathcal{A}^{\leq 1} \]

and
\[ F_\star X \in \mathcal{A}^{\leq 0} \iff F_\star X \in \prod (\mathcal{A}^{\geq 1})_\mathcal{A} \]
\[ \iff \text{Hom}_\mathcal{A} (Y, F_\star X) = 0 \text{ for all } Y \in \mathcal{A}^{\geq 1} \]
\[ \iff \text{Hom}_\mathcal{D} (F Y, X) = 0 \text{ for all } Y \in \mathcal{A}^{\leq 1}. \]

If \( F \) is the inclusion functor, then the above conditions are equivalent to the following conditions
\[ \text{Hom}_\mathcal{D} (X, Y) = 0 \text{ for all } Y \in \mathcal{A}^{\leq 0} \]
and
\[ \text{Hom}_\mathcal{D} (Y, X) = 0 \text{ for all } Y \in \mathcal{A}^{\geq 0}, \]
respectively. \( \square \)

### 4 Perverse t-structures

Let \( \mathcal{D}, \mathcal{A} \) and \( \mathcal{B} \) be triangulated categories and let \( F : \mathcal{A} \rightarrow \mathcal{D} \) and \( G : \mathcal{D} \rightarrow \mathcal{B} \) be exact functors. In particular, we assume that \( \mathcal{A} \) is the kernel of \( G \) and \( F \) is the inclusion functor.

**Lemma 4.1.** We have the following.

(i) If \( G \) admits a left adjoint functor \( G_L \), then \( \text{Ker}(G) = (G_L \mathcal{B})^{\perp} \).

(ii) If \( G \) admits a right adjoint functor \( G_R \), then \( \text{Ker}(G) = \perp (G_R \mathcal{B}) \).

**Proof.** By Yoneda Lemma, we obtain
\[ \text{Ker}(G) = \{ X \in \mathcal{D} \mid GX \cong 0 \} = \{ X \in \mathcal{D} \mid \text{Hom}_\mathcal{B} (Y, GX) = 0 \text{ for all } Y \in \mathcal{B} \} = \{ X \in \mathcal{D} \mid \text{Hom}_\mathcal{D} (G_L Y, X) = 0 \text{ for all } Y \in \mathcal{B} \} = (G_L \mathcal{B})^{\perp} \]
if \( G \) admits a left adjoint functor \( G_L \), and
\[ \text{Ker}(G) = \{ X \in \mathcal{D} \mid GX \cong 0 \} = \{ X \in \mathcal{D} \mid \text{Hom}_\mathcal{B} (GX, Y) = 0 \text{ for all } Y \in \mathcal{B} \} = \{ X \in \mathcal{D} \mid \text{Hom}_\mathcal{D} (X, G_R Y) = 0 \text{ for all } Y \in \mathcal{B} \} = \perp (G_R \mathcal{B}) \]
if $G$ admits a right adjoint functor $G_R$, respectively.

**Proposition 4.2.** We assume that the following conditions hold:

(i) $G$ admits a left adjoint functor $GL$ and a right adjoint functor $GR$,

(ii) the functors $G_l$ and $G_r$ are both fully faithful.

Then the above conditions $GL1 \to GL5$ hold.

**Proof.** We verify the above conditions $GL1 \to GL5$.

(GL1) Since $G_l$ is fully faithful, we have the exact triple $G_l B \overset{\iota}{\to} \mathcal{D} \overset{Q_l}{\to} \mathcal{D}/G_l B$, where $G_l B$ is an essential image of $G_l$. By Proposition 5.8 of [Ter 16] and Lemma 4.1, the functor $p_l := Q_l \circ F: \mathcal{A} \to \mathcal{D}/G_l B$ is an exact equivalence and the Verdier localization functor $Q_l$ has a right adjoint functor $(Q_l)^{\circ} := F \circ p_l^{-1}$, where $p_l^{-1}$ is the quasi-inverse functor to $p_l$. Put $F_l := p_l^{-1} \circ Q_l$. Then we have

$$\text{Hom}_A (F_l X, Y) = \text{Hom}_A (p_l^{-1} Q_l, X, Y)$$

for any $X \in \mathcal{D}$ and any $Y \in \mathcal{A}$. Hence $F_l$ is a left adjoint functor to $F$.

Similarly, since $G_r$ is fully faithful, we have the exact triple $G_r B \overset{\iota}{\to} \mathcal{D} \overset{Q_r}{\to} \mathcal{D}/G_r B$, where $G_r B$ is an essential image of $G_r$. By Proposition 5.9 of [Ter 16] and Lemma 4.1, the functor $p_r := Q_r \circ F: \mathcal{A} \to \mathcal{D}/G_r B$ is an exact equivalence and the Verdier localization functor $Q_r$ has a right adjoint functor $(Q_r)^{\circ} := F \circ p_r^{-1}$, where $p_r^{-1}$ is the quasi-inverse functor to $p_r$. Put $F_r := p_r^{-1} \circ Q_r$. Then we have

$$\text{Hom}_A (X, F_r Y) = \text{Hom}_A (X, p_r^{-1} Q_r Y)$$

for any $X \in \mathcal{A}$ and any $Y \in \mathcal{D}$. Hence $F_r$ is a right adjoint functor to $F$.

(GL2) This follows from the assumption $G_l \circ G \circ G_r$.

(GL3) This follows from the construction of the functor $F$.

(GL4) Since $G$ is a right adjoint functor to $G_l$, by Proposition 5.8 of [Ter 16], we have the distinguished triangle

$$X_{G_l B} \to X \to X_{(G_l B)^{\circ}} \to TX_{G_l B}.$$ 

By Lemma 4.1. (i), we have $(G_l B)^{\circ} \simeq \text{Ker}(G) \simeq F \mathcal{A}$. There exist an object $Y \in \mathcal{B}$ with $X_{G_l B} = G_l Y$ and an object $Z \in \mathcal{A}$ with $X_{(G_l B)^{\circ}} \simeq F Z$. Hence we obtain the distinguished triangle

$$G_l Y \to X \to F Z \to TG_l Y.$$ 

(4.1)

Then, applying $G$ to (4.1), we have

$$GG_l Y \to GX \to GF Z \to TGG_l Y.$$
Since $GG_l Y \simeq Y$ by Proposition 2.8. (i) of [Ter 16] and $GFZ = 0$, we see that $Y \simeq GX$. Moreover, applying $F_l$ to (4.1), we have

$$F_l G_l Y \Rightarrow F_l X \Rightarrow F_l FZ \Rightarrow TF_l G_l Y.$$ 

Since $F_l G_l Y = 0$ by Proposition 3.1. (i) and $F_l FZ \simeq Z$ by Proposition 3.1. (iii), we see that $Z \simeq F_l X$. Therefore, by Proposition 3.1. (ii), we obtain the distinguished triangle

$$G_l GX \Rightarrow X \Rightarrow FF_l X \Rightarrow TG_l GX.$$ 

Similarly, since $G$ is a left adjoint functor to $G_r$, by Proposition 5.9 of [Ter 16], we have the distinguished triangle

$$X_{-(G_r B)} \Rightarrow X \Rightarrow X_{G_r B} \Rightarrow TX_{-(G_r B)}.$$ 

By Lemma 4.1. (ii), we have $-(G, B) = \text{Ker}(G) \simeq FA$. There exist an object $V \in \mathcal{A}$ with $FV = X_{-(G_r B)}$ and an object $W \in \mathcal{B}$ with $G_r W \simeq X_{G_r B}$. Hence we obtain the distinguished triangle

$$(4.2) \quad FV \Rightarrow X \Rightarrow G_r W \Rightarrow T FV.$$ 

Then, applying $G$ to (4.2), we have

$$GFV \Rightarrow GX \Rightarrow G G_r W \Rightarrow T G F V.$$ 

Since $GFV = 0$ and $G G_r W \simeq W$ by Proposition 2.8. (ii) of [Ter 16], we see that $W \simeq GX$. Moreover, applying $F_r$ to (4.2), we have

$$F_r F V \Rightarrow F_r X \Rightarrow F_r G_r W \Rightarrow T F_r F V.$$ 

Since $F_r F V \simeq V$ by Proposition 3.1. (iii) and $F_r G_r W = 0$ by Proposition 3.1. (i), we see that $V \simeq F_r X$. Therefore, by Proposition 3.1. (ii), we obtain the distinguished triangle

$$F F_r X \Rightarrow X \Rightarrow G_r GX \Rightarrow T F F_r X.$$ 

(GL 5) $F$ is fully faithful because it is the inclusion functor. $G_l$ and $G_r$ are fully faithful from the assumption. \(\square\)

**Definition 4.3.** Let $\mathcal{D}$ and $\mathcal{B}$ be triangulated categories with $t$-structures $(\mathcal{D}^{<0}, \mathcal{D}^{>0})$ on $\mathcal{D}$ and $(\mathcal{B}^{<0}, \mathcal{B}^{>0})$ on $\mathcal{B}$, respectively. Let $G : \mathcal{D} \rightarrow \mathcal{B}$ be a $t$-exact functor and let $\mathcal{A}$ be the kernel of $G$. Let $(\mathcal{A}^{\leq 0}, \mathcal{A}^{>0}) := (\mathcal{A} \cap \mathcal{D}^{<0}, \mathcal{A} \cap \mathcal{D}^{>0})$ be the induced $t$-structure on $\mathcal{A}$. Assume that $G$ admits a left adjoint functor and a right adjoint functor, and the adjoint functors are both fully faithful. For an integer $p$, we obtain a new $t$-structure $(\mathcal{D}^{<0}, \mathcal{D}^{>0})$ on $\mathcal{D}$ by glueing $t$-structures $(\mathcal{A}^{\leq p}, \mathcal{A}^{>p})$ and $(\mathcal{B}^{<0}, \mathcal{B}^{>0})$ from Theorem 3.2 and Proposition 4.2, where

- $\mathcal{D}^{<0} := \{X \in \mathcal{D} \mid GX \in \mathcal{B}^{<0} \text{ and } \text{Hom}_{\mathcal{D}}(X, Y) = 0 \text{ for all } Y \in \mathcal{A}^{>0}\}$,
- $\mathcal{D}^{>0} := \{X \in \mathcal{D} \mid GX \in \mathcal{B}^{>0} \text{ and } \text{Hom}_{\mathcal{D}}(Y, X) = 0 \text{ for all } Y \in \mathcal{A}^{>0}\}$.

Then the t-structure $(\mathcal{D}^{<0}, \mathcal{D}^{>0})$ on $\mathcal{D}$ is called the perverse t-structure with a perversity $p$ obtained by glueing $t$-structures $(\mathcal{A}^{\leq p}, \mathcal{A}^{>p})$ and $(\mathcal{B}^{<0}, \mathcal{B}^{>0})$ and its heart is the abelian category

$$\text{Per}(\mathcal{D}/\mathcal{B}) := \mathcal{D}^{<0} \cap \mathcal{D}^{>0}.$$ 

In particular, when $p = -1$, we put
and objects of \( \Per(D/B) \) are called perverse objects.

The following example is from algebraic geometry. For a projective variety \( X \) over the complex numbers \( \mathbb{C} \), we denote by \( D(X) \) the derived category of complexes of quasi-coherent \( \mathcal{O}_X \)-modules with coherent cohomology sheaves. The derived category \( D(X) \) has the standard \( t \)-structure.

**Example 4.4.** ([Bri 02]) Let \( f : Y \rightarrow X \) be a birational morphism of projective varieties such that \( Rf_*=\mathcal{O}_Y=\mathcal{O}_X \). By Grothendieck-Verglaire duality theorem, the functor \( Rf_* : D(Y) \rightarrow D(X) \) has the left adjoint functor \( Lf^* \) and the right adjoint functor \( f^! \). Note that the functor \( Lf^* \) is fully faithful, and then the functor \( f^! \) is also fully faithful, because the dualizing functors \( D_x \) and \( D_v \) are autoequivalences: \( f^! = D_v \circ Lf^* \circ D_x \).

Now let us write \( \mathcal{D} = D(Y) \), \( \mathcal{B} = D(X) \) and \( G = Rf_* \), and then we put \( \mathcal{A} = \Ker(G) \). Then we are in the situation of Definition 4.3. A complex \( F^* \in D(Y) \) lies in \( \Ker(Rf_*) \) if and only if its cohomology sheaves \( H^i(f^*) \) lies in \( \Ker(Rf_*) \). For a complex

\[ F^* : \cdots \rightarrow F^{-2} \rightarrow F^{-1} \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots \]

of \( \Ker(Rf_*) \), we see that the truncation complex

\[ \tau^{D(Y)}_{D_0}(F^*) : \cdots \rightarrow F^{-2} \rightarrow F^{-1} \rightarrow \Ker(d_0) \rightarrow 0 \rightarrow \cdots \]

belongs to \( \Ker(Rf_*) \). We have the induced \( t \)-structure \( (\Ker(Rf_*))^{<0}, (\Ker(Rf_*))^{\geq 0} \) on \( \Ker(Rf_*) \) from the standard \( t \)-structure \( (D(Y))^{<0}, (D(Y))^{\geq 0} \) on \( D(Y) \) by Proposition 2.5. Hence we obtain the perverse \( t \)-structure \( (D(Y))^{<0}, (D(Y))^{\geq 0} \) on \( D(Y) \) with a perversity \( p \) by glueing the induced \( t \)-structure \( (\Ker(Rf_*))^{<0}, (\Ker(Rf_*))^{\geq 0} \) on \( \Ker(Rf_*) \) and the standard \( t \)-structure \( (D(X))^{<0}, (D(X))^{\geq 0} \) on \( D(X) \). Then objects of \( \Per(Y/X) := \Per(D(Y)/D(X)) \) are called per-verse cohernt sheaves.

**References**


[Huy 06] D. Huybrechts, *Fourier–Mukai transforms in algebraic geometry*, Oxford Math-


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