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Triangulated Categories. I: Foundations

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要旨

三角圏は1977年に J-L. Verdier により定義され、近年、代数幾何学、代数的位相幾何学、表現論、代数解析学など数学の多くの分野において、その有効性に注目が集まっている。この研究ノートは、主に代数幾何学に現れる三角圏や導来圏の性質についてまとめた一連の研究の最初の部分である。

第２節では第３節において必要となる圏と関手に関する最小限の知識をまとめてある。三角圏の定義とその基本性質を第３節で解説する。また、これらの研究の成果として得られた論文（Masahiro Ohno and Hiroyuki Terakawa, A spectral sequence and nef vector bundles of the first Chern class two on hyperquadrics, ANNALI DELL'UNIVERSITA' DI FERRARA, Published online: 11 September 2013.）の理解に必要となる基礎概念も解説する。

1　Introduction

Triangulated categories are now very popular tool in algebraic geometry. This note is the first part of our study of triangulated categories in algebraic geometry. We shall explain the definition and fundamental properties of a triangulated category and describe their proofs in detail.

2　Categories and functors

In this section, we introduce basic notions of categories and functors. Main references are [ML98], [HS97] and [KS06].

Definition 2.1.　A **category** $\mathcal{C}$ consists of the following data:

(i) a set $\mathrm{Ob}(\mathcal{C})$, whose elements are called the *objects* of $\mathcal{C}$,

(ii) for each ordered pair of objects X, Y of $\mathrm{Ob}(\mathcal{C})$, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, whose elements are called *morphisms* from X to Y,

(iii) for each ordered triple of objects X, Y, Z of $\mathrm{Ob}(\mathcal{C})$, a map

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z),$$

called the *composition* map and denoted by $(f, g) \mapsto g \circ f$.

These data satisfy the following conditions:

1. the sets $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $\mathrm{Hom}_{\mathcal{C}}(X', Y')$ are disjoint unless $X = X'$ and $Y = Y'$,
2. the composition $\circ$ is associative, i.e., for $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$ and $h \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$, we have
$$(h \circ g) \circ f = h \circ (g \circ f),$$
3. for each $X \in \mathrm{Ob}(\mathcal{C})$, there exists the *identity morphism* $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ such that
$$f \circ \mathrm{id}_X = f \text{ and } \mathrm{id}_X \circ g = g$$
for all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and all $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$. □

We often write $X \in \mathcal{C}$ instead of $X \in \mathrm{Ob}(\mathcal{C})$, and $f : X \to Y$ to denote a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. The set of all morphisms in $\mathcal{C}$ is denoted by $\mathrm{Mor}(\mathcal{C})$.

Definition 2.2. (*i*) A morphism $f : X \to Y$ is an **isomorphism** if there exists $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. If $f : X \to Y$ is an isomorphism, we write $f : X \xrightarrow{\sim} Y$ or else $X \simeq Y$.

(*ii*) A morphism $f : X \to Y$ is a **monomorphism** if for any two morphisms $g_1, g_2 : Z \to X$, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.

(*iii*) A morphism $f : X \to Y$ is an **epimorphism** if for any two morphisms $g_1, g_2 : Y \to Z$, $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$. □

Definition 2.3. Let $\mathcal{C}$ be a category.

1. A **subcategory** $\mathcal{C}'$ of $\mathcal{C}$, denoted by $\mathcal{C}' \subset \mathcal{C}$, is a category $\mathcal{C}'$ such that $\mathrm{Ob}(\mathcal{C}') \subset \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}'}(X, Y) \subset \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for all $X, Y \in \mathcal{C}'$, with the induced composition map, and the identity morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}'}(X, X)$ for all $X \in \mathcal{C}'$.
2. A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is called a **full subcategory** if $\mathrm{Hom}_{\mathcal{C}'}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for all $X, Y \in \mathcal{C}'$.
3. A full subcategory $\mathcal{C}'$ of $\mathcal{C}$ is called a **strictly full subcategory** of $\mathcal{C}$ if it is closed under isomorphisms, i.e., $X \in \mathcal{C}$ belongs to $\mathcal{C}'$ whenever X is isomorphic to an object of $\mathcal{C}'$.

□

Definition 2.4. Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories. A **functor** F from $\mathcal{C}$ to $\mathcal{C}'$ consists of the following data and rules:

(i) a map $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}')$,

(ii) a map $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))$ for all $X, Y \in \mathcal{C}$.

These data satisfy the following conditions:

$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \mathcal{C}$,

$F(g \circ f) = F(g) \circ F(f)$ for all $f : X \to Y$, $g : Y \to Z$. □

Definition 2.5. Let $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{C}''$ be categories and let $F : \mathcal{C} \to \mathcal{C}'$, $G : \mathcal{C}' \to \mathcal{C}''$ be functors. The **composition** $G \circ F : \mathcal{C} \to \mathcal{C}''$ is a functor defined by
$$(G \circ F)(X) := G(F(X))$$

for all $X \in \mathrm{Ob}(\mathcal{C})$ and

$$(G \circ F)(f) := G(F(f))$$

for all $f \in \mathrm{Mor}(\mathcal{C})$. □

Definition 2.6. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a functor. We say that F is **faithful** (resp. **full**, **fully faithful**) if the map

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))$$

is injective (resp. surjective, bijective) for all $X, Y \in \mathcal{C}$. □

Definition 2.7. A functor $F\colon \mathcal{C} \to \mathcal{C}$ is an **automorphism** of $\mathcal{C}$ if there exists a functor $G\colon \mathcal{C} \to \mathcal{C}$ such that $F \circ G = G \circ F = \mathrm{id}_{\mathcal{C}}$. In this case, we write F^{-1} instead of G. □

Definition 2.8. Let $\mathcal{C}$, $\mathcal{C}'$ be categories and let $F_1, F_2\colon \mathcal{C} \to \mathcal{C}'$ be functors. A **morphism of functors** $\theta\colon F_1 \to F_2$ consists of a morphism $\theta_X : F_1(X) \to F_2(X)$ for all $X \in \mathcal{C}$ such that for all $f : X \to Y$, the diagram:

$$\begin{array}{ccc} F_1(X) & \xrightarrow{\theta_X} & F_2(X) \\ {\scriptstyle F_1(f)}\downarrow & & \downarrow{\scriptstyle F_2(f)} \\ F_1(Y) & \xrightarrow{\theta_Y} & F_2(Y) \end{array}$$

commutes. □

Definition 2.9. Let $\mathcal{C}$ be a category. Let I be a set and $\{X_i\}_{i \in I}$ a family of objects in $\mathcal{C}$. A **product** of that family is a pair $(P, (p_i)_{i \in I})$ where

(i) P is an object of $\mathcal{C}$,

(ii) for every $i \in I$, $p_i : P \to X_i$ is a morphism of $\mathcal{C}$,

and this pair satisfies the following property: for every other pair $(Q, (q_i)_{i \in I})$ where

(1) Q is an object of $\mathcal{C}$,

(2) for every $i \in I$, $q_i : Q \to X_i$ is a morphism of $\mathcal{C}$,

there exists a unique morphism $r : Q \to P$ such that for every index i, $q_i = p_i \circ r$. □

We shall write $\prod_{i \in I} X_i$ for the product of a family $\{X_i\}_{i \in I}$.

Definition 2.10. Let $\mathcal{C}$ be a category. Let I be a set and $\{X_i\}_{i \in I}$ a family of objects in $\mathcal{C}$. A **coproduct** of that family is a pair $(P, (s_i)_{i \in I})$ where

(i) P is an object of $\mathcal{C}$,

(ii) for every $i \in I$, $s_i : X_i \to P$ is a morphism of $\mathcal{C}$,

and this pair satisfies the following property: for every other pair $(Q, (t_i)_{i \in I})$ where

(1) Q is an object of $\mathcal{C}$,

(2) for every $i \in I$, $t_i : X_i \to Q$ is a morphism of $\mathcal{C}$,

there exists a unique morphism $u : P \to Q$ such that for every index i, $t_i = u \circ s_i$. □

We shall write $\coprod_{i \in I} X_i$ for the coproduct of a family $\{X_i\}_{i \in I}$.

Proposition 2.11. *We have isomorphisms, functorial with respect to $Y \in \mathcal{C}$:*

(*i*) $\mathrm{Hom}_{\mathcal{C}}(\coprod_{i \in I} X_i, Y) \simeq \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(X_i, Y)$,

(*ii*) $\mathrm{Hom}_{\mathcal{C}}(Y, \prod_{i \in I} X_i) \simeq \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(Y, X_i)$.

Proof. This follows from the definitions. □

Definition 2.12. (*i*) An **additive category** is a category $\mathcal{C}$ satisfying the following axioms AD1−AD3:

AD1 There exists a *zero object* $0 \in \mathrm{Ob}(\mathcal{C})$, i.e. an object such that $\mathrm{Hom}_{\mathcal{C}}(0, 0)$ is the zero group.

AD2 For all $X, Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has a structure of an additive group, and the composition map $\circ$ is bilinear.

AD3 For all $X, Y \in \mathcal{C}$, there exist an object $Z \in \mathcal{C}$ and morphisms

$$i_1 : X \to Z,\ i_2 : Y \to Z,\ p_1 : Z \to X,\ p_2 : Z \to Y,$$

such that

$$p_2 \circ i_1 = 0,\ p_1 \circ i_2 = 0,\ p_1 \circ i_1 = \mathrm{id}_X,\ p_2 \circ i_2 = \mathrm{id}_Y$$

and

$$i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{id}_Z.$$

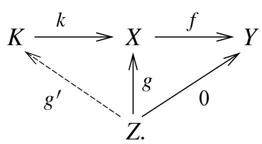
Such an object Z is called the **direct sum** of X and Y, and denoted by $X \oplus Y$.

(*ii*) Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor of additive categories. F is said to be **additive** if the map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))$ is additive for any $X, Y \in \mathcal{C}$. □

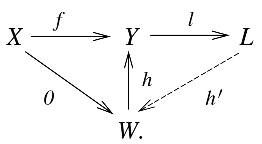
If $\{X_i\}_{i \in I}$ is a family of objects of an additive category $\mathcal{C}$ and the coproduct $\coprod_{i \in I} X_i$ exists in $\mathcal{C}$, it is denoted by $\oplus_{i \in I} X_i$ and called the **direct sum** of the X_i's.

Definition 2.13. Let $\mathcal{C}$ be an additive category and let $f : X \to Y$ be a morphism in $\mathcal{C}$.

(*i*) We say that $k : K \to X$ is a **kernel** of f if $f \circ k = 0$ and for any morphism $g : Z \to X$ with $f \circ g = 0$, there exists a unique morphism $g' : Z \to K$ such that $g = k \circ g'$. This is visualized by the diagram:



(ii) We say that $k : Y \to L$ is a **cokernel** of f if $k \circ f = 0$ and for any morphism $h : Y \to W$ with $h \circ f = 0$, there exists a unique morphism $h' : L \to W$ such that $h = h' \circ l$. This is visualized by the diagram:



□

Definition 2.14. An **abelian category** is an additive category $\mathcal{A}$ satisfying the following axioms AB1 and AB2:

AB1 Every morphism in A has a kernel and a cokernel,

AB2 Every monomorphism is a kernel, and every epimorphism is a cokerel. □

3 Triangulated categories

In this section, we introduce the notion of a triangulated category and prove its fundamental properties. Main references are [Ver77], [GM03] and [KS06].

Let $\mathcal{D}$ be an additive category and let T be an additive automorphism of $\mathcal{D}$.

Definition 3.1. A **triangle** in $\mathcal{D}$ is a sequence of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX.$$

A **morphism** of triangles is a commutative diagram:

(3.1)
$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & TX \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle T(\alpha)} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & TX'. \end{array}$$

An **isomorphism** of triangles is a commutative diagram (3.1) such that α, β and γ are isomorphisms in $\mathcal{D}$. □

Definition 3.2. A **triangulated category** is an additive category $\mathcal{D}$ with an additive automorphism $T : \mathcal{D} \to \mathcal{D}$, called the **translation functor**, endowed with a family of triangles, called **distinguished triangles**, satisfying the following axioms TR0—TR5:

TR0 A triangle isomorphic to a distinguished triangle is a distinguished triangle.

TR1 The triangle $X \xrightarrow{\mathrm{id}_X} X \to 0 \to TX$ is a distinguished triangle.

TR2 For all $f : X \to Y$, there exists a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow TX.$$

The object Z is called a **cone** of the morphism f, which is denoted by $\mathrm{Cone}(f)$.

TR3 A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ is a distinguished triangle if and only if $Y \xrightarrow{g} Z \xrightarrow{h} TX \xrightarrow{-T(f)} TY$ is a distinguished triangle.

TR4 Given two distinguished triangles $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} TX'$ and morphisms $\alpha : X \to X'$ and $\beta : Y \to Y'$ with $f' \circ \alpha = \beta \circ f$, there exists a morphism $\gamma : Z \to Z'$ giving rise to a morphism of distinguished triangles:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & TX \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle T(\alpha)} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & TX'. \end{array}$$

TR5　Given three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{h} Z' \longrightarrow TX,$$

$$Y \xrightarrow{g} Z \xrightarrow{k} X' \longrightarrow TY,$$

$$X \xrightarrow{g \circ f} Z \xrightarrow{l} Y' \longrightarrow TX,$$

there exists a distinguished triangle $Z' \xrightarrow{u} Y' \xrightarrow{v} X' \xrightarrow{w} TZ'$ making the diagram below commutative:

(3.2)

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{h} & Z' & \longrightarrow & TX \\ \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle u} & & \| \\ X & \xrightarrow{g \circ f} & Z & \xrightarrow{l} & Y' & \longrightarrow & TX \\ \downarrow{\scriptstyle f} & & \| & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle T(f)} \\ Y & \xrightarrow{g} & Z & \xrightarrow{k} & X' & \longrightarrow & TY \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle l} & & \| & & \downarrow{\scriptstyle T(h)} \\ Z' & \dashrightarrow & Y' & \overset{v}{\dashrightarrow} & X' & \overset{w}{\dashrightarrow} & TZ'. \end{array}$$

Diagram (3.2) is called the **octahedron diagram**.

Definition 3.3.　Let $(\mathcal{D}, T)$ and $(\mathcal{D}', T')$ be triangulated categories. An additive functor $F : \mathcal{D} \to \mathcal{D}'$ is called an **exact** functor if the following two conditions are satisfied:

(i)　There exists an isomorphism of functors

$$F \circ T \simeq T' \circ F.$$

(ii)　Any distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow TX$$

in $\mathcal{D}$ is mapped to a distinguished triangle

$$F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow TF(X)$$

in $\mathcal{D}'$, where $F(TX)$ is identified with $TF(X)$ via the isomorphism of functors in (i). □

Lemma 3.4.　*Let* $(\mathcal{D}, T)$ *be a triangulated category. Let* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} TX'$ *be two distinguished triangles.*

(i)　*For morphisms* $\beta : Y \longrightarrow Y'$ *and* $\gamma : Z \longrightarrow Z'$ *with* $g' \circ \beta = \gamma \circ g$, *there exists a morphism* $\alpha : X \longrightarrow X'$ *giving rise to a morphism of distinguished triangles*:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & TX \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \gamma} & & \Big\downarrow{\scriptstyle T(\alpha)} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & TX'. \end{array}$$

(*ii*) *For morphisms* $\gamma : Z \longrightarrow Z'$ *and* $T(\alpha) : TX \longrightarrow TX'$ *with* $f' \circ \gamma = T(\alpha) \circ h$, *there exists a morphism* $\beta : Y \longrightarrow Y'$ *giving rise to a morphism of distinguished triangles*:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & TX \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \gamma} & & \Big\downarrow{\scriptstyle T(\alpha)} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & TX'. \end{array}$$

Proof. This follows from TR3 and TR4. □

Proposition 3.5. *If* $X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow TX$ *is a distinguished triangle, then* $g \circ f = 0$.

Proof. From TR1 and TR4, we obtain a morphism of distinguished triangle:

$$\begin{array}{ccccccc} X & \xrightarrow{\mathrm{id}_X} & X & \longrightarrow & 0 & \longrightarrow & TX \\ \Big\downarrow{\scriptstyle \mathrm{id}_X} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}_{TX}} \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & TX. \end{array}$$

Hence $g \circ f = 0$.

Definition 3.6. A full subcategory $\mathcal{C}$ of a triangulated category $\mathcal{D}$ is said to be **extension-closed** or **closed under extensions** if, $Y \in \mathcal{D}$ belongs to $\mathcal{C}$ whenever $X \longrightarrow Y \longrightarrow Z \longrightarrow TX$ is a distinguished triangle in $\mathcal{D}$ with $X \in \mathcal{C}$ and $Z \in \mathcal{C}$. The extension-closed subcategory of $\mathcal{D}$ **generated by a full subcategory** $\mathcal{S} \subset \mathcal{D}$ is the smallest extension-closed full subcategory of $\mathcal{D}$ containing $\mathcal{S}$.

Definition 3.7. Let $(\mathcal{D}, T)$ be a triangulated category and let $\mathcal{A}$ be an abelian category.

1. Assume $F : \mathcal{D} \to \mathcal{A}$ is a covariant additive functor. Then F is called a **covariant cohomological** functor if for any distinguished triangle

   $$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

   in $\mathcal{D}$, the sequence

   $$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z)$$

   is exact in $\mathcal{A}$.
2. Assume $F : \mathcal{D} \to \mathcal{A}$ is a contravariant additive functor. Then F is called a **contravariant cohomological** functor if for any distinguished triangle

   $$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

in $\mathcal{D}$, the sequence

$$F(Z) \xrightarrow{F(v)} F(Y) \xrightarrow{F(u)} F(X)$$

is exact in $\mathcal{A}$.

3. For a cohomological functor F, we define $F^n := F \circ T^n$ for each $n \in \mathbb{Z}$. □

Let $X \to Y \to Z \to TX$ be any distinguished triangle in $\mathcal{D}$. If F is a covariant cohomological functor, then by the axiom TR3, we obtain a long exact sequence

$$\cdots \to F^{n-1}(Z) \to F^n(X) \to F^n(Y) \to F^n(Z) \to F^{n+1}(X) \to \cdots.$$

Similarly, If F is a cotravariant cohomological functor, then we obtain a long exact sequence

$$\cdots \to F^{n+1}(X) \to F^n(Z) \to F^n(Y) \to F^n(X) \to F^{n-1}(Z) \to \cdots.$$

Proposition 3.8. *For any $W \in \mathcal{D}$, the two functors* $\mathrm{Hom}_{\mathcal{D}}(W, \bullet)$ *and* $\mathrm{Hom}_{\mathcal{D}}(\bullet, W)$ *are cohomological.*

Proof. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \to TX$ be a distinguished triangle and let $W \in \mathcal{D}$. For $\varphi \in \mathrm{Hom}(W, Y)$ with $g \circ \varphi = 0$, it follows from Lemma 3.4 that we have a morphism $\psi : W \to X$ such that

$$\begin{array}{ccccccc} W & \xrightarrow{\mathrm{id}_W} & W & \longrightarrow & 0 & \longrightarrow & TW \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi} & & \downarrow & & \downarrow{\scriptstyle \mathrm{T}(\psi)} \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & TX. \end{array}$$

is a morphism of distinguished triangles. Hence $\varphi = f \circ \psi$. Therefore we obtain an exact sequence

$$\mathrm{Hom}(W, X) \xrightarrow{f \circ} \mathrm{Hom}(W, Y) \xrightarrow{g \circ} \mathrm{Hom}(W, Z).$$

Similarly, for $\varphi \in \mathrm{Hom}(Y, W)$ with $\varphi \circ f = 0$, it follows from Lemma 3.4 that we have a morphism $\psi : Z \to W$ such that

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & TW \\ \downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} & & \downarrow \\ 0 & \longrightarrow & W & \xrightarrow{\mathrm{id}_W} & W & \longrightarrow & 0. \end{array}$$

is a morphism of distinguished triangles. Hence $\varphi = \psi \circ g$. Therefore we obtain an exact sequence

$$\mathrm{Hom}(Z, W) \xrightarrow{\circ g} \mathrm{Hom}(Y, W) \xrightarrow{\circ f} \mathrm{Hom}(X, W).$$

This completes the proof. □

Proposition 3.9. *Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} TX'$ be distinguished triangles in a triangulated category $\mathcal{D}$, and let $g : Y \to Y'$ be a morphism. Then the following conditions are equivalent.*

(*a*) $v' \circ g \circ u = 0$,

(b) *there exists a morphism* $f : X \to X'$ *such that* $u' \circ f = g \circ u$,

(b') *there exists a morphism* $h : Z \to Z'$ *such that* $v' \circ g = h \circ v$,

(c) *there exists a morphism of triangles* (f, g, h).

Moreover, if the conditions are satisfied and $\mathrm{Hom}_{\mathcal{D}}(X, T^{-1}Z') = 0$, *then the morphism* f (*resp*. h) *of* (b) (*resp*. (b')) *is unique*.

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle T(f)} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & TX' \end{array}$$

Proof. (a) $\Rightarrow$ (b). Applying the functor $\mathrm{Hom}(X, \bullet)$ to the second distinguished triangle, we have a long exact sequence

$$\cdots \to \mathrm{Hom}(X, T^{-1}Z') \to \mathrm{Hom}(X, X') \to \mathrm{Hom}(X, Y') \to \mathrm{Hom}(X, Z') \to \cdots.$$

Since $g \circ u \in \mathrm{Hom}(X, Y')$, we obtain the required result from the sequence. And we have the uniqueness of f if $\mathrm{Hom}(X, T^{-1}Z') = 0$.

(b) $\Rightarrow$ (a). $v' \circ g \circ u = v' \circ u \circ f = 0$.

(a) $\Rightarrow$ (b'). Applying the functor $\mathrm{Hom}(\bullet, Z')$ to the first distinguished triangle, we have a long exact sequence.

$$\cdots \to \mathrm{Hom}(TX, Z') \to \mathrm{Hom}(Z, Z') \to \mathrm{Hom}(Y, Z') \to \mathrm{Hom}(X, Z') \to \cdots.$$

Since $v' \circ g \in \mathrm{Hom}(Y, Z')$, we obtain the required result from the sequence. And we have the uniqueness of h if $\mathrm{Hom}(TX, Z') = \mathrm{Hom}(X, T^{-1}Z') = 0$.

(b') $\Rightarrow$ (a) $v' \circ g \circ u = h \circ v \circ u = 0$.

(b) $\Rightarrow$ (c). This follows from Axiom TR4.

(c) $\Rightarrow$ (a). $v' \circ g \circ u = h \circ v \circ u = 0$. □

Corollary 3.10. *Let* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ *be a distinguished triangle in a triangulated category* $\mathcal{D}$. *Suppose* $\mathrm{Hom}_{\mathcal{D}}(X, T^{-1}Z) = 0$. *Then we have*

(i) *the cone of* u *is unique up to unique isomorphism*.

(ii) w *is a unique morphism* $x : Z \to TX$ *such that the triangle* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{x} TX$ *is distinguished*.

Proof. If in Proposition 3.9, $X = X'$, $Y = Y'$ and f, g are identities, then Z is isomorphic to Z', so $\mathrm{Hom}(X, T^{-1}Z') = 0$, and ($i$) is the result of the uniqueness of h. For (ii), we apply Proposition 3.9 to the following diagram.

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_Y} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \mathrm{id}_{T(X)}} \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{x} & TX \end{array}$$

Then we have $h = \mathrm{id}_Z$ by the uniqueness of h. Therefore $w = x$. □

Proposition 3.11. *Let $(\mathcal{D}, T)$ be a triangulated category. Consider a morphism of distinguished triangles*:

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & TX \\ \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \downarrow T(\alpha) \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & TX'. \end{array}$$

If two of the morphisms, α, β and γ are isomorphisms, then so is the third.

Proof. For any $W \in \mathcal{D}$, we obtain the commutative diagram:

$$\begin{array}{ccccccccccc} \cdots \to & \mathrm{Hom}(W,X) & \to & \mathrm{Hom}(W,Y) & \to & \mathrm{Hom}(W,Z) & \to & \mathrm{Hom}(W,TX) & \to \cdots \\ & \alpha_*\downarrow & & \beta_*\downarrow & & \gamma_*\downarrow & & \downarrow T(\alpha)_* & \\ \cdots \to & \mathrm{Hom}(W,X') & \to & \mathrm{Hom}(W,Y') & \to & \mathrm{Hom}(W,Z') & \to & \mathrm{Hom}(W,TX') & \to \cdots. \end{array}$$

Since $\mathrm{Hom}(W, \bullet)$ is cohomological, the rows are long exact sequences. If two of the morphisms α, β and γ be isomorphisms, then the corresponding morphisms in the above diagram are also isomorphisms, and thus so is the third. Hence we obtain the required isomorphism from the Yoneda lemma. □

Corollary 3.12. *Let $X \xrightarrow{f} Y \to Z \to TX$ be a distinguished triangle in a triangulated category. Then $f: X \to Y$ is an isomorphism if and only if Z is isomorphic to* 0.

Proof. Consider the morphism of distinguished triangles:

$$\begin{array}{ccccccc} X & \xrightarrow{\mathrm{id}_X} & X & \longrightarrow & 0 & \longrightarrow & TX \\ \mathrm{id}_X\downarrow & & \downarrow f & & \downarrow & & \downarrow \mathrm{id}_{TX} \\ X & \xrightarrow[f]{} & Y & \longrightarrow & Z & \longrightarrow & TX. \end{array}$$

Then we obtain the statement from Proposition 3.11. □

Proposition 3.13. *Let $(\mathcal{D}, T)$ be a triangulated category which admits direct sums indexed by a set I.*

(*i*) *Let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{D}$. Then $T(\oplus_{i\in I} X_i) \simeq \oplus_{i\in I} TX_i$.*

(*ii*) *Let $\{X_i \to Y_i \to Z_i \to TX_i\}_{i\in I}$ be a family of distinguished triangles in $\mathcal{D}$. Then*

$$\oplus_{i\in I} X_i \to \oplus_{i\in I} Y_i \to \oplus_{i\in I} Z_i \to \oplus_{i\in I} TX_i$$

is a distinguished triangle.

Proof. (*i*) Let $W \in \mathcal{D}$. Then we have

$$\begin{aligned} \mathrm{Hom}(T(\oplus_{i\in I} X_i), W) &\simeq \mathrm{Hom}(\oplus_{i\in I} X_i, T^{-1}W) \\ &\simeq \prod_{i\in I} \mathrm{Hom}(X_i, T^{-1}W) \\ &\simeq \prod_{i\in I} \mathrm{Hom}(TX_i, W) \\ &\simeq \mathrm{Hom}(\oplus_{i\in I} TX_i, W). \end{aligned}$$

Hence $T(\oplus_{i\in I} X_i) \simeq \oplus_{i\in I} TX_i$.

(*ii*) By the axiom TR2, we have an object $Z \in \mathcal{D}$ such that

$$\oplus_{i\in I} X_i \to \oplus_{i\in I} Y_i \to Z \to T(\oplus_{i\in I} X_i)$$

is a distinguished triangle. By the axiom TR 3, there exist morphisms of distinguished triangles

$$\begin{array}{ccccccc} X_i & \longrightarrow & Y_i & \longrightarrow & Z_i & \longrightarrow & TX_i \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \oplus_{i\in I} X_i & \longrightarrow & \oplus_{i\in I} Y_i & \longrightarrow & Z & \longrightarrow & T(\oplus_{i\in I} X_i). \end{array}$$

and they induce a morphism of triangles

$$\begin{array}{ccccccc} \oplus_{i\in I} X_i & \longrightarrow & \oplus_{i\in I} Y_i & \longrightarrow & \oplus_{i\in I} Z_i & \longrightarrow & \oplus_{i\in I} TX_i \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \oplus_{i\in I} X_i & \longrightarrow & \oplus_{i\in I} Y_i & \longrightarrow & Z & \longrightarrow & T(\oplus_{i\in I} X_i). \end{array}$$

Let $W \in \mathcal{D}$. Apply $\mathrm{Hom}(\bullet, W)$, and we obtain the commutative diagram of complexes

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{D}}(T(\oplus_{i\in I} Y_i), W) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(T(\oplus_{i\in I} X_i), W) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(Z, W) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{D}}(\oplus_{i\in I} TY_i, W) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(\oplus_{i\in I} TX_i, W) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(\oplus_{i\in I} Z_i, W) \end{array}$$

$$\begin{array}{ccccc} \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(\oplus_{i\in I} Y_i, W) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(\oplus_{i\in I} X_i, W) \\ & \downarrow & & \downarrow \\ \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(\oplus_{i\in I} Y_i, W) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(\oplus_{i\in I} X_i, W). \end{array}$$

The first row is exact since the functor $\mathrm{Hom}_{\mathcal{D}}$ is cohomological. The second row is isomorphic to

$$\prod_{i\in I} \mathrm{Hom}_{\mathcal{D}}(TY_i, W) \to \prod_{i\in I} \mathrm{Hom}_{\mathcal{D}}(TX_i, W) \to \prod_{i\in I} \mathrm{Hom}_{\mathcal{D}}(Z_i, W)$$
$$\to \prod_{i\in I} \mathrm{Hom}_{\mathcal{D}}(Y_i, W) \to \prod_{i\in I} \mathrm{Hom}_{\mathcal{D}}(X_i, W).$$

Since the functor $\prod_{i\in I}$ is exact on the category of abelian groups, this complex is exact. Since the vertical arrows except the middle one are isomorphisms, the middle one is an isomorphism by the five lemma. □

Since $X \xrightarrow{\mathrm{id}_X} X \to 0 \to TX$ and $0 \to Z \xrightarrow{\mathrm{id}_Z} Z \to 0$ are distinguished triangles, by Proposition 3.13, we have the following distinguished triangle:

$$X \xrightarrow{\begin{bmatrix}\mathrm{id}_X \\ 0\end{bmatrix}} X \oplus Z \xrightarrow{[0\ \mathrm{id}_Z]} Z \xrightarrow{0} TX. \tag{3.3}$$

Conversely we obtain the following.

Proposition 3.14. *Let $X \to Y \to Z \xrightarrow{h} TX$ be a distinguished triangle in a triangulated category. If $h = 0$, then this triangle is isomorphic to the distinguished triangle* (3.3).

Proof. From Lemma 3.4, we have a morphism of distinguished triangles

$$
\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \xrightarrow{0} & TX \\
{\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_Z} & & \downarrow{\scriptstyle \mathrm{id}_{TX}} \\
X & \xrightarrow[\begin{bmatrix}\mathrm{id}_X \\ 0\end{bmatrix}]{} & X \oplus Z & \xrightarrow[{[0\ \mathrm{id}_Z]}]{} & Z & \xrightarrow[0]{} & TX.
\end{array}
$$

Then by Proposition 3.11. (ii) , we obtain an isomorphism $Y \simeq X \oplus Z$. □

Proposition 3.15. *Let $(\mathcal{D}, T)$ and $(\mathcal{D}', T')$ be triangulated categories and let $F : \mathcal{D}' \to \mathcal{D}$ be a fully faithful exact functor. Then a triartgle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T'X$ is distinguished if and only if the triangle $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{F(h)} TF(X)$ is distinguished.*

Proof. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T'X$ be a triangle in $\mathcal{D}'$, and assume that $FX \xrightarrow{F(f)} FY \xrightarrow{F(g)} FZ \xrightarrow{F(h)} T(FX)$ is a distinguished triangle in $\mathcal{D}$. We have a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g'} Z' \xrightarrow{h'} T'X$ in $\mathcal{D}'$. Then by TR3 and Proposition 3.11, we obtain an isomorphism of distinguished triangles in $\mathcal{D}$:

$$
\begin{array}{ccccccc}
F(X) & \xrightarrow{F(f)} & F(Y) & \xrightarrow{F(g)} & F(Z) & \xrightarrow{F(h)} & TF(X) \\
\| & & \| & & \downarrow\simeq & & \| \\
F(X) & \xrightarrow[F(f)]{} & F(Y) & \xrightarrow[F(g')]{} & F(Z') & \xrightarrow[F(h')]{} & TF(X).
\end{array}
$$

Therefore the triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T'X$ is distinguished.

Definition 3.16. A subcategory $\mathcal{D}'$ of a triangulated category $(\mathcal{D}, T)$ is a **triangulated subcategory** if $(\mathcal{D}', T|_{\mathcal{D}'})$ admits the structure of a triangulated category and the inclusion functor $\iota : \mathcal{D}' \to \mathcal{D}$ is exact. □

Definition 3.17. A full subcategory $\mathcal{C}$ of a triangulated category $(\mathcal{D}, T)$ is said to be **closed under translation** if $T\mathcal{C} = \mathcal{C}$. □

Corollary 3.18. *Let $\mathcal{D}'$ be a full triangulated subcategory of a triangulated category $(\mathcal{D}, T)$. If a triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow TX$ in $\mathcal{D}'$ is distinguished in $\mathcal{D}$, then it is distinguished in $\mathcal{D}'$.* □

Corollary 3.19. *Let $(\mathcal{D}, T)$ be a triangulated category and* let *$\mathcal{D}'$ be a full subcategory of $\mathcal{D}$. Then $\mathcal{D}'$ is a triartgulated subcategory if and oniy if the following conditions are satisfied:*

- (i) *$\mathcal{D}'$ is closed under translation,*
- (ii) *for any distinguished triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow TX$ in $\mathcal{D}$ with $X, Y \in \mathcal{D}'$, the object Z is isomorphic to an object in $\mathcal{D}'$.*

In particular, strictly full triangulated subcategories are extension-closed.

Proof. Let $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow TX$ in $\mathcal{D}'$ be a distinguished triangle in $\mathcal{D}'$. Then we have a distinguished triangle $X \xrightarrow{f} Y \longrightarrow \mathrm{Cone}(f) \longrightarrow TX$ in $\mathcal{D}$. By the condition (ii), there exists an object $Z' \in \mathcal{D}'$ such that $Z' \simeq \mathrm{Cone}\,(f)$, and thus we obtain a commutative diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & TX \\ \| & & \| & & \downarrow{\scriptstyle \simeq} & & \| \\ X & \xrightarrow[f]{} & Y & \longrightarrow & Z' & \longrightarrow & TX. \end{array}$$

Hence the distinguished triangle $X \xrightarrow{f} Y \to Z \to TX$ in $\mathcal{D}'$ is also distinguished in $\mathcal{D}$. □

Proposition 3.20. *In a triangulated category $\mathcal{D}$, consider the diagram of solid arrows*

$$\begin{array}{ccccccc} X^0 & \xrightarrow{u} & X^1 & \xrightarrow{v} & X^2 & \xrightarrow{w} & TX^0 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} & & \downarrow & & \downarrow{\scriptstyle T(f)} \\ Y^0 & \xrightarrow{u'} & Y^1 & \longrightarrow & Y^2 & \longrightarrow & TY^0 \\ {\scriptstyle g}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle T(g)} \\ Z^0 & \dashrightarrow & Z^1 & \dashrightarrow & Z^2 & \dashrightarrow & TZ^0 \\ {\scriptstyle h}\downarrow & & \downarrow & & \downarrow & \text{ac} & \downarrow{\scriptstyle -T(h)} \\ TX^0 & \xrightarrow[T(u)]{} & TX^1 & \xrightarrow[T(v)]{} & TX^2 & \xrightarrow[-T(w)]{} & T^2X^0. \end{array}$$

Assume that the two first rows and columns are distinguished triangles. Then the dotted arrows can be completed in order that all squares are commutative, except the one labeled "ac" which is anti-commutative, and all rows and all columns are distinguished triangles. See [BBD82].

Proof. By Axiom TR5, we have the following three (octahedron) diagrams.

(3.4)

$$\begin{array}{ccccccc} X^0 & \xrightarrow{f} & Y^0 & \xrightarrow{g} & Z^0 & \xrightarrow{h} & TX^0 \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathrm{id}} \\ X^0 & \xrightarrow{u'\circ f} & Y^1 & \xrightarrow{\varphi} & W & \xrightarrow{\psi} & TX^0 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle T(f)} \\ Y^0 & \xrightarrow{u'} & Y^1 & \xrightarrow{v'} & Y^2 & \xrightarrow{w'} & TY^0 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle T(g)} \\ Z^0 & \overset{\alpha}{\dashrightarrow} & W & \overset{\beta}{\dashrightarrow} & Y^2 & \overset{\varepsilon}{\dashrightarrow} & TZ^0. \end{array}$$

(3.5)

$$\begin{array}{ccccccc} X^0 & \xrightarrow{u} & X^1 & \xrightarrow{v} & X^2 & \xrightarrow{w} & TX^0 \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \mathrm{id}} \\ X^0 & \xrightarrow{f'\circ u} & Y^1 & \xrightarrow{\varphi} & W & \xrightarrow{\psi} & TX^0 \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle T(u)} \\ X^1 & \xrightarrow{f'} & Y^1 & \xrightarrow{d'} & Z^1 & \xrightarrow{h'} & TX^1 \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle T(v)} \\ X^2 & \overset{\gamma}{\dashrightarrow} & W & \overset{\delta}{\dashrightarrow} & Z^1 & \overset{\zeta}{\dashrightarrow} & TX^2. \end{array}$$

(3.6)

$$\begin{array}{ccccccc}
X^2 & \xrightarrow{\gamma} & W & \xrightarrow{\delta} & Z^1 & \xrightarrow{\zeta} & TX^2 \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle v''} & & \downarrow{\scriptstyle \mathrm{id}} \\
Y^2 & \xrightarrow{f''=\beta\circ\gamma} & Y^2 & \xrightarrow{g''} & Z^2 & \xrightarrow{h''} & TX^2 \\
\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle w''} & & \downarrow{\scriptstyle T(\gamma)} \\
W & \xrightarrow{\beta} & Y^2 & \xrightarrow{\varepsilon} & TZ^0 & \xrightarrow{-T(\alpha)} & TW \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle g''} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle T(\delta)} \\
Z^1 & \xrightarrow{v''} & Z^2 & \xrightarrow{w''} & TZ^0 & \xrightarrow{-T(u'')} & TZ^1.
\end{array}$$

Note that we have $\delta\circ\alpha = u''$ from the lower right-hand square in the diagram(3.6). It follows from these diagrams that we obtain distinguished triangles $X^2 \to Y^2 \to Z^2 \to TX^2$ and $Z^0 \to Z^1 \to Z^2 \to TZ^0$. Moreover we see that the squares (1)-(8) are commutative and the square (9) is anti-commutative as follows.

(3.7)

$$\begin{array}{ccccccc}
X^0 & \xrightarrow{u} & X^1 & \xrightarrow{v} & X^2 & \xrightarrow{w} & TX^0 \\
\downarrow{\scriptstyle f} & (1) & \downarrow{\scriptstyle f'} & (2) & \downarrow{\scriptstyle f''} & (3) & \downarrow{\scriptstyle T(f)} \\
Y^0 & \xrightarrow{u'} & Y^1 & \xrightarrow{v'} & Y^2 & \xrightarrow{w'} & TY^0 \\
\downarrow{\scriptstyle g} & (4) & \downarrow{\scriptstyle g'} & (5) & \downarrow{\scriptstyle g''} & (6) & \downarrow{\scriptstyle T(g)} \\
Z^0 & \xrightarrow{u''} & Z^1 & \xrightarrow{v''} & Z^2 & \xrightarrow{w''} & TZ^0 \\
\downarrow{\scriptstyle h} & (7) & \downarrow{\scriptstyle h'} & (8) & \downarrow{\scriptstyle h''} & (9) & \downarrow{\scriptstyle -T(h)} \\
TX^0 & \xrightarrow[T(u)]{} & TX^1 & \xrightarrow[T(v)]{} & TX^2 & \xrightarrow[-T(w)]{} & T^2X^0.
\end{array}$$

(1) $f'\circ u = u'\circ f$. This is an assumption.

(2) $f''\circ v = \beta\circ\gamma\circ v = \beta\circ\varphi\circ f' = v'\circ f'$.

(3) $T(f)\circ w = T(f)\circ\psi\circ\gamma = w'\circ\beta\circ\gamma = w'\circ f''$.

(4) $g'\circ u' = \delta\circ\varphi\circ u' = \delta\circ\alpha\circ g = u''\circ g$.

(5) $g''\circ v' = g''\circ\beta\circ\varphi = v''\circ\delta\circ\varphi = v''\circ g'$.

(6) $T(g)\circ w' = \varepsilon = w''\circ g''$.

(7) $h'\circ u'' = h'\circ\delta\circ\alpha = T(u)\circ\psi\circ\alpha = T(u)\circ h$.

(8) $h''\circ v'' = \zeta = T(v)\circ h'$.

(9) $-T(h)\circ w'' = -T(\psi\circ\alpha)\circ w'' = -T(\psi)\circ T(\alpha)\circ w'' = T(\psi)\circ(-T(\alpha))\circ w'' =$
$T(\psi)\circ T(\gamma)\circ h'' = T(\psi\circ\gamma)\circ h'' = T(w)\circ h'' = -(-T(w)\circ h'')$. □

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